Lorentz Covariant Formulation for Force in SR

The case that prompted the idea of rewriting the laws of physics in covariant form:

$$\frac{dp}{dt} = F \text{ where } F = m_0 \frac{d^2 x}{dt^2} \tag{1.1}$$

In one dimension, the relativistic momentum is simply $p = m_0\gamma(v)v$, so:

$$\frac{dp}{dt} = m_0\gamma^3 \frac{dv}{dt} \tag{1.2}$$

In three dimensions, things get a little more complicated:

$$\mathbf{p} = m_0\gamma(v)\mathbf{v} \tag{1.3}$$

$$\frac{d\mathbf{p}}{dt} = m_0\gamma\left(\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{v\gamma^2}{c^2}\frac{dv}{dt}\right) \tag{1.4}$$

If $\frac{d\mathbf{v}}{dt}$ and $\mathbf{v}$ have the same direction and sense, then:

$$\frac{d\mathbf{p}}{dt} = m_0\gamma^3 \frac{d\mathbf{v}}{dt} \tag{1.5}$$

Returning to the one-dimensional case, in frame S', moving with speed V along the common x-axis:

$$p' = m_0\gamma(v')v' \tag{1.6}$$

$$v' = \frac{v - V}{1 - \frac{vV}{c^2}} \tag{1.7}$$

$$\gamma(v') = \gamma(v)\gamma(V)\left(1 - \frac{vV}{c^2}\right) \tag{1.8}$$

$$\frac{dp'}{dt'} = m_0\gamma^3(v)\frac{dv}{dt} \tag{1.9}$$

So, $\frac{dp'}{dt'}=\frac{dp}{dt}$. While p is frame variant, it turns out that for the one dimensional case $\frac{dp}{dt}$ is frame invariant. Another interesting observation is that in frame S Newton's law is $\frac{dp}{dt}=m_0\gamma^3(v)\frac{d^2x}{dt^2}$ then in frame S' the law has the same form: $\frac{dp'}{dt'}=m_0\gamma^3(v')\frac{d^2x'}{dt'^2}$.
This is an immediate consequence of the fact that:

$$\frac{d^2x'}{dt'^2}=\frac{dv'}{dt'}=\frac{dv'}{dt}\frac{dt}{dt'}=\frac{1}{\gamma(V)(1-\frac{vV}{c^2})}\frac{d}{dt}\left(\frac{v-V}{1-\frac{vV}{c^2}}\right)=\frac{\frac{dv}{dt}}{\gamma^3(V)(1-\frac{vV}{c^2})^3} \qquad (1.10)$$

so:

$$\gamma^3(v')\frac{d^2x'}{dt'^2}=\gamma^3(v)\frac{d^2x}{dt^2} \qquad (1.11)$$

We can conclude that, at relativistic speeds, Newton law $\frac{dp}{dt}=m_0\frac{d^2x}{dt^2}$ must be replaced by $\frac{dp}{dt}=m_0\gamma^3(v)\frac{d^2x}{dt^2}$ in order to be covariant. In other words, the Newtonian expression of force $F=m_0\frac{d^2x}{dt^2}$ needs to be replaced with its relativistic counterpart $F=m_0\gamma^3(v)\frac{d^2x}{dt^2}$. At low speeds $\gamma(v)\approx 1$ so the two expressions become indistinguishable but at relativistic speeds this is no longer the case.

The above prompted Minkowski to reformulate Newton's law as:

$$\frac{d\mathbf{p}}{d\tau}=\mathbf{F_M} \qquad (1.12)$$

where

$d\tau=dt\sqrt{1-(\frac{v}{c})^2}$ is called "proper time interval"

$\mathbf{F_M}=\frac{\mathbf{F}}{\sqrt{1-(\frac{v}{c})^2}}$ is called the "Minkowski force". So, $\frac{d\mathbf{p}}{d\tau}=\mathbf{F_M}$ is identical with $\frac{d\mathbf{p}}{dt}=\mathbf{F}$.

We can now introduce the 4-vector $\tilde{\mathbf{p}}=(p_x,p_y,p_z,W/c)=(p_x,p_y,p_z,p_w)$ where W is the total energy. The 4-vector $\tilde{\mathbf{p}}$ transforms exactly the same way as (x,y,z,ct):

$$p_x' = \gamma(V)(p_x - \frac{V}{c}p_w)$$
$$p_y' = p_y$$
$$p_z' = p_z \qquad (1.13)$$
$$p_w' = \gamma(V)(p_w - \frac{V}{c}p_x)$$

It follows immediately that $\frac{d\tilde{\mathbf{p}}}{d\tau} = (\frac{dp_x}{d\tau}, \frac{dp_y}{d\tau}, \frac{dp_z}{d\tau}, \frac{d(W/c)}{d\tau})$ is also a 4-vector, since it transforms the same way as (x, y, z, ct) as well. Indeed:

$$\frac{dp_x'}{d\tau} = \gamma(V)(\frac{dp_x}{d\tau} - \frac{V}{c}\frac{d(W/c)}{d\tau})$$
$$\frac{d(W'/c)}{d\tau} = \gamma(V)(\frac{d(W/c)}{d\tau} - \frac{V}{c}\frac{dp_x}{d\tau}) \qquad (1.14)$$

The above gave the idea of introducing a new construct[5]: $\tilde{\mathbf{F}} = (\mathbf{F}_M, \mathbf{F}_M.\mathbf{v}/c)$. Since the left hand side of the equation $\frac{d\tilde{\mathbf{p}}}{d\tau} = \tilde{\mathbf{F}}$ is a 4-vector, the equation will be satisfied only for $\tilde{\mathbf{F}} = (\mathbf{F}_M, \mathbf{F}_M.\mathbf{v}/c)$ quantities that are 4-vectors themselves.

Wave Equation Covariance

Let's start with the Lorentz transforms:

$$x' = \gamma(V)(x - Vt)$$
$$t' = \gamma(V)(t - \frac{V}{c^2}x)$$
$$y' = y \qquad (2.1)$$
$$z' = z$$

In frame S, the wave equation is (in 1+1 space):

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\frac{\partial^2 \mathbf{H}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \mathrm{H}}{\partial t^2} \qquad (2.2)$$

From (2.1) we see that:

$$
\begin{aligned}
&\frac{\partial}{\partial x}=\frac{\partial}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial}{\partial t'}\frac{\partial t'}{\partial x}=\gamma(\frac{\partial}{\partial x'}-\frac{V}{c^2}\frac{\partial}{\partial t'})\\
&\frac{\partial}{\partial t}=\frac{\partial}{\partial x'}\frac{\partial x'}{\partial t}+\frac{\partial}{\partial t'}\frac{\partial t'}{\partial t}=\gamma(\frac{\partial}{\partial t'}-V\frac{\partial}{\partial x'})\\
&\frac{\partial^2}{\partial x^2}=\gamma^2(\frac{\partial^2}{\partial x'^2}-2\frac{V}{c^2}\frac{\partial^2}{\partial x'\partial t'}+\frac{V^2}{c^4}\frac{\partial^2}{\partial t'^2})\\
&\frac{\partial^2}{\partial t^2}=\gamma^2(\frac{\partial^2}{\partial x'^2}-2V\frac{\partial^2}{\partial x'\partial t'}+\frac{\partial^2}{\partial t'^2})
\end{aligned}
\tag{2.3}
$$

On the other hand:

$$
\begin{aligned}
&E_x=E_x^{'}\\
&E_y=\gamma(E_y^{'}+\frac{V}{c}H_z^{'})\\
&E_z=\gamma(E_z^{'}-\frac{V}{c}H_y^{'})\\
&H_x=H_x^{'}\\
&H_y=\gamma(H_y^{'}-\frac{V}{c}E_z^{'})\\
&H_z=\gamma(H_z^{'}+\frac{V}{c}E_y^{'})
\end{aligned}
\tag{2.4}
$$

Substituting the last two operators from (2.3) and (2.4) into (2.1) we obtain, after some reduction of like terms:

$$
\begin{aligned}
&\frac{\partial^2\mathbf{E'}}{\partial x^{'2}}=\frac{1}{c^2}\frac{\partial^2\mathbf{E'}}{\partial t^{'2}}\\
&\frac{\partial^2\mathbf{H'}}{\partial x^{'2}}=\frac{1}{c^2}\frac{\partial^2\mathbf{H'}}{\partial t^{'2}}
\end{aligned}
\tag{2.5}
$$

i.e. the Lorentz transforms preserve the wave equation form.

The Lorentz-invariance of the Maxwell equations can be expressed in the coincise form as:

$$
\begin{aligned}
&F_{\nu\mu}=\frac{\partial\phi_\nu}{\partial x^\mu}-\frac{\partial\phi_\mu}{\partial x^\nu}\\
&\frac{\partial F^{\mu\nu}}{\partial x^\nu}=J^\mu
\end{aligned}
\tag{2.6}
$$

where

$$\phi^{\mu} = (A_x, A_y, A_z, \phi)$$
$$J^{\mu} = \rho_0 \frac{dx^{\mu}}{ds} = (\rho \frac{u_x}{c}, \rho \frac{u_y}{c}, \rho \frac{u_z}{c}, \rho) \tag{2.7}$$

are four-vectors.

General Covariance

In order to obtain the general covariant form for equations (2.6) we need to replace the ordinary differentiation in (2.6) with covariant differentiation. The four-vectors defined by (2.7) are covariant by virtue of their definition.

$$F_{\nu\mu} = (\phi_{\mu})_{\nu} - (\phi_{\nu})_{\mu} = \frac{\partial \phi_{\nu}}{\partial x^{\mu}} - \frac{\partial \phi_{\mu}}{\partial x^{\nu}}$$
$$(F^{\mu\nu})_{\nu} = J^{\mu} \tag{3.1}$$

The first equation does not even change form with respect to (2.6) owing to the fact that the terms in the Christoffel symbols cancel out. We can prove easily that the definition (3.1) produces indeed a form that is generally covariant. Indeed. on one hand:

$$F'_{\nu\mu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} F_{\alpha\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} (\frac{\partial \phi_{\alpha}}{\partial x^{\beta}} - \frac{\partial \phi_{\beta}}{\partial x^{\alpha}}) = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial \phi_{\alpha}}{\partial x'^{\nu}} - \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial \phi_{\beta}}{\partial x'^{\mu}} \tag{3.2}$$

On the other hand:

$$\phi'_{\mu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \phi_{\alpha}$$
$$\phi'_{\nu} = \frac{\partial x^{\beta}}{\partial x'^{\nu}} \phi_{\beta} \tag{3.3}$$

so:

$$\frac{\partial \phi'_{\mu}}{\partial x'^{\nu}} - \frac{\partial \phi'_{\nu}}{\partial x'^{\mu}} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial \phi_{\alpha}}{\partial x'^{\nu}} - \frac{\partial x^{\beta}}{\partial x'^{\nu}} \frac{\partial \phi_{\beta}}{\partial x'^{\mu}} \tag{3.4}$$

Therefore:

$$F'_{\mu\nu} = \frac{\partial \phi'_{\mu}}{\partial x'^{\nu}} - \frac{\partial \phi'_{\nu}}{\partial x'^{\mu}} \tag{3.5}$$

Though we have demonstrated the general covariance of the rank two tensor F , this does not mean anything in terms of the usefulness of the transforms. Indeed, in the case of Shubert's crackpot theory, the partial derivatives $\frac{\partial x^{\alpha}}{\partial x'^{\mu}}, \frac{\partial x^{\beta}}{\partial x'^{\nu}}$ acting on $F_{\alpha\beta}$ are virtually meaningless owing to their recursive form , so, for all practical purposes $F'_{\nu\mu}$ cannot be evaluated anywhere. This is a very serious defect, in addition to the fact that the jacobian of the transformation may be degenerate over a whole spacetime domain (an infinity of (x,t) points). In fact, the determinant of the jacobian needs to be equal to unity in order to preserve the metric invariance [1], something that does not happen in the case of Shubert's jacobian since it is a function of the arbitrary synchronization functions S_i and S_j. So, all the crap put up by Guest254 and AlphaNumeric is just that: crap.

[1] C. Moller , The Theory of Relativity (pp. 94-95)